EVALUATING PROOFS AND CONJECTURES CONSTRUCTED BY PRE-SERVICE MATHEMATICS TEACHERS

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This study focused on investigating the ability of 58 pre-service mathematics teachers' (PSMTs) to construct-evaluate mathematical conjectures-proofs in a mathematics course. The combined construction-evaluation activity of conjectures-and-proofs helps illuminate pre-service mathematics teachers' understanding of proof. The result of the study demonstrated that the number of instances where the PSMTs constructed conjectures were less than the number of instances where they constructed arguments to prove/disprove assertions during the semester. Additionally, the PSMTs usually constructed conjectures when they were explicitly asked to do so. The majority of the arguments that were constructed by the PSMTs attempted to provide an explanation for why the assertion held true which may show that the explanatory role of arguments indeed held an essential criterion for the PSMTs.

Keywords: Teacher Education-Preservice

Proof is viewed as a cornerstone of mathematics and an essential element for developing deep understanding (e.g., Ball & Bass, 2000; NCTM, 2000). Yet, research indicates that students of all levels tend to have limited understanding of proof and struggle with constructing proofs (e.g. Harel & Sowder, 1998). Many researchers demonstrated that the empirical reasoning is pervasive among school students including advanced or high-attaining secondary students (e.g. Coe & Ruthven, 1994; Healy & Hoyles, 2000), university students including mathematics majors (e.g., Goetting, 1995) as well as prospective and in-service teachers (e.g. Morris, 2002; Simon &Bume, 1996).

Despite the importance of teachers' understanding of proof, relatively little research has investigated aspects of prospective or practicing teachers' understanding of proofs (Goetting, 1995; Morris, 2002; Stylianides, Stylianides, & Philippou, 2007). Furthermore, previous studies focus solely on teachers' understanding of the distinction between deductive and empirical arguments by asking them to evaluate researcher generated arguments. Stylianides and Stylianides (2009) criticized that there has been limited research about how instructions can help pre-service teachers' develop their understanding of proof. Thus, this study aims to contribute to literature on pre-service teachers' understanding of proof by reporting on pre-service mathematics teachers' (PSMTs)processes of constructing-evaluating mathematical conjectures-proofs during a course in which PSMTs specifically engaged in proving tasks.

Functions of Proof in Classrooms

Traditionally the function of proof has been seen almost exclusively as being to verify or justify the correctness of mathematical statements (e.g. Ball & Bass, 2000). The "verification" function of proof is often interpreted in subjective terms, establishing the truth of a statement with an individual's belief in the truth of a statement and thus allocating proof a role in the subjective acquisition of such belief. However, as Bell (1976) argues, proof is not necessarily a prerequisite for conviction; proof is essentially a social activity of validation or establishing results, which follows reaching a conviction. Duval (2002) argues that a proof can change the logical value as well as epistemic value of a statement. That is, a proof may logically validate a statement, but it can also affect the belief of the cognizing subject as to the truth of the statement. These two functions of

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proof—to convince individuals and to establish results in the field—are by no means the only functions of proof in mathematical activity.

Researchers have contributed to such elaboration on the functions of proof both by reflecting on its many roles in the discipline of mathematics and by identifying its roles in mathematical understanding. These roles are identified by de Villiers by building on the work of others (Balacheff, 1988; Bell, 1976; Hanna, 1990; Hersh, 1993) as follows: (a) verification (concerned with the truth of a statement), (b) explanation (providing insight into why it is true), (c) systematization (the organization of various results into a deductive system of axioms, major concepts and theorems), (d) discovery (invention of new results), (e) communication (the transmission of mathematical knowledge), and (f) intellectual challenge (de Villiers, 1990, p.18).

Hersh (1993) argues that the role of proof in the classroom and the role of proof in mathematical discipline could be different, stating that the purpose of proof in mathematical discipline to be to convince, while in a classroom it should be to explain. Knuth (2002) has echoed this theme, arguing to teachers that proofs are valuable because they can help students understand mathematics. Hanna (1990) distinguishes between “proofs that prove” and “proofs that explain”.Thus, the development of proofs in the course where the study took place served two related functions: (a) as means for explaining why an assertion was true or false by showing how the statement of a theorem coheres and connects with the key properties of the concepts involved in the proof, which will be referred as Type P_1proof and (b) as a means for justifying that an assertion was true thereby promoting conviction, which will be referred as Type P_2 proof in the study.

Methodology

In this section, the context in which the research reported here took place, the research participants and the data collection and analysis processes will be described.

Participants

Participants of the study were 58 pre-service mathematics teachers (PSMTs) who are certified to teach mathematics in grade 5- 8. The PSMTs enrolled in a mathematics course during the semester of spring of 2016. The course was worth three university credits, and so the class met 3 hours per week for a semester. The course was designed to cover a wide range of mathematical topics in three major mathematical domains (algebra, geometry and number theory). The PSMTs were offered various opportunities to engage with mathematical proofs including constructing-evaluating proofs, representing them in different ways (using everyday language, algebra, or pictures), and examining the correspondences among different representations.

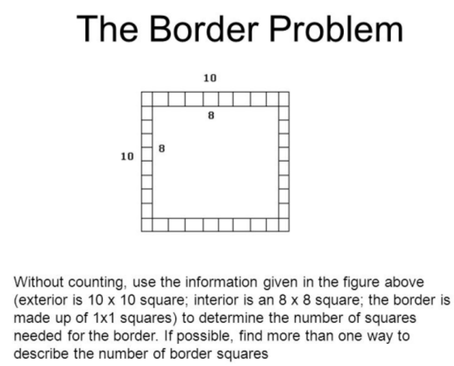
Tasks

A sample of proof tasks in which PSMTs were engaged in during the semester will be presented here in order for readers to better conceptualize PSMTs’ conjecture/proof construction and evaluation processes (see Table 1).

Data Collection Process

The participants were engaged in a course where they were required to work in groups of 6. The participants were engaged in solving tasks that were adopted from existing literature (see Table 1). All instructions were videotaped during three hours of the instruction time for 14 weeks in the semester. The video camera was located at the corner of the classroom where the board was captured. These videos served as the main data source for the study. In addition to the class videos, the PSMTs’ written responses to some of the tasks and their class assignments were also collected.

Table 1: Sample of the Tasks

Task A was adopted from Wilburne (2014). The task was as follow: A fast food restaurant sells chicken nuggets in packs of 4 and 7. What is the largest number of nuggets you cannot buy? How do you know this is the largest number you cannot buy?	**Task B** was adopted from (Weber, 2003). The task was as follows: For every odd integer n, $n^2 - 1$ is divisible by 8.
Task C was as follows: Justify that the area formula of a kite is $\frac{d_1 d_2}{2}$, where d_1 and d_2 are the diagonals of the kite.	**Task D** was adopted from Boaler and Humphreys (2005). The task was as follows: 

Data Analysis

The data analysis started with reviewing the videos of the instructions first. After the first review of the videos, the parts where the PSMTs were engaged in construction-and-evaluation of mathematical conjectures-and-proofs were selected and transcribed. Later, the selected segments and the transcript of these segments were viewed again and the PSMTs' proof constructions were coded in one of the following categories: Type P_1: valid general argument that explains why an assertion was true by standing of the underlying mathematical concepts, Type P_2: valid general argument that proves that an assertion was true but did not provide any insight into why it might hold true, Type P_3: general argument that fall short of being acceptable proofs , and Type P_4: unsuccessful attempt for a valid general argument (invalid, unfinished, or irrelevant responses (or potentially relevant response but the relevance was not made evident). Categories TypeP_1 through TypeP_4 represent four different arguments constructed by the PSMTs in decreasing levels of sophistication (from a mathematical stand point), with Type P_4 representing the least sophisticated argument. The construction of conjectures was coded in one of the following categories: Type C_1: conjecture that was constructed as a response to a requested wish (usually by the instructor) in a given context, Type C_2: conjecture that was constructed spontaneously as a natural extension of a task, and TypeC_3: incorrect conjectures. As opposed to categories for proofs, categories Type C_1 through Type C_3 for coding PSMTs' conjectures were not listed in hierarchical levels of sophistication.

Results

General Findings

Table 2 summarizes the distribution of proof-conjecture constructions during the semester. As it was evident in the table, the majority of the proofs constructed during the class were Type P_1 proof, valid general argument that explains why an assertion was true or false by showing how the statement of a theorem coheres and connects with the key properties of the concepts involved in the proof. Of the remaining 29 proving occurrences, 18 of them were Type P_2 proofs, valid general arguments that established that an assertion was true thereby promoted conviction, but provided little or no explanation for why it held true.

Type P_4 proofs, unsuccessful attempt for a valid general argument (i.e. incorrect, invalid, unfinished, or irrelevant responses-or potentially relevant response but the relevance was not made evident in the argument), were proposed 8 times during the semester; however, it should be noted that the PSMTs were aware of the limitations of these arguments. Therefore, they were able to evaluate those arguments as not proofs or as not correct argument during the class discussions. Of these 8 unsuccessful attempts to prove the class tasks, 2 arguments were empirical arguments. The PSMTs who proposed these empirical arguments as well as the others in the class were able to state the fact that generalizing from specific cases was not a valid mode of argumentation.

The number of the cases where conjectures were constructed during the class happened significantly less than the number of cases where proofs were proposed (13 vs. 57). Additionally, the majority of the cases where the conjectures were constructed occurred as a response to a request made usually by the instructor (Type C_1 conjecture). Incorrect conjectures were proposed 3 times during the instructions and after these conjectures were proposed the other PSMTs in the class were able to refute these conjectures by providing a valid counterexample.

Table 2: Distribution of Proof-Conjecture Constructions During the Class

Proofs				Conjectures		
Type P_1	Type P_2	Type P_3	Type P_4	Type C_1	Type C_2	Type C_3
28 (49%)	18 (31%)	3 (5%)	8 (14%)	6 (46%)	4 (30%)	3 (23%)

Classroom Episodes That Represent Different Types of Proof-Conjecture Constructions

In this part two classroom episodes will be shared to exemplify some of the codes used to codify the PSMTs' proof –and-conjecture conjectures.

Episode 1: Chicken tender task. In this episode, the PSMTs were engaged in working on chicken tender task in their groups.

Orhan: Our group has decided that the numbers that can be represented as 28k+27, where k is an integer, cannot be bought in the packets of 7 and/or 4.
Instructor: Ok. Where did 28 and 27 come from?
Orhan: 7 times 4 is 28, so 28 can be bought in packets of either 7 or 4 or its multiples.
Instructor: OK, if k=1 then how many nuggets do you think you cannot buy, umm, 55?
Orhan: Yes
Instructor: Can we have 55 nuggets in the packets of 7 and/or 4?

Merve: Yes. We can have 5 packets of 7s and 5 packets of 4s. So, we can get 55 nuggetsin total.

Instructor: Ok, so the numbers that are represented as 28k+27 can be bought in packets of 7 and 4. Anybody else have an argument? Selman?

Selman: (Writing numbers on the board). You can represent all numbers by adding 4. For instance, if you add 4 to 11, you will get 15; if you add 4 to 12 you will get 16; if you add 4 to 14 you will get 18 and it will continue like this. These numbers cannot be bought in packets of 7 or 4 (highlighting the numbers underlined below). Umm, I needed to check 21 because it is to add 4 to 17 and I know 17 cannot be represented as addends of 4 (or multiples of 4) and 7 (or multiples of 7). However, I have found that 21 is 3 times 7, so it is okay too. Now you will continue this pattern for all numbers 21+4=25, 22+4=26,23+4=27,24+4=28...etc.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30

Ayse: When you get modulo 7, the residue classes will be 1,2,3,5,or 6 (4 cannot be counted here because we can get the packets of four). When you get modulo 4, the residue classes will be 1,2,and 3. When you add all residue classes up you will get the answer-17.

In this episode, the conjecture proposed by Orhan was coded as Type C_3: incorrect conjecture. The instructor posed a counterexample as a necessary condition for the realization of falsity of the conjecture. The PSMTs were able to explain how the example 55 contradicted the conjecture and refuted Orhan's conjecture. Selman's argument was coded as Type P_1 since it was built on the properties of numbers. It was a correct argument to justify that 17 was the highest number of chicken tenders that could not be bought in packets of 4 and/or 7. Ayse identified all residue classes of modulo 7 except 4 (since it could be a possible answer) and added them up to reach the answer of 17. However, her argument did not include a justification for the assertion that the residues would always be the highest number that could not be bought in the packets of 4 or 7. Indeed, when her argument was applied to different numbers such as packets of 6 and 4, it would not give the correct response. Therefore, her argument was coded as Type P_4.

Episode 2: Area perimeter task. In this episode, the PSMTs were engaged in working on geometry task-investigating the relationship between area and perimeter of rectangles. The instructor asked the PSMTs to construct conjectures about area and perimeter of rectangles. Cihat proposed the following conjecture: "With *the same perimeter, the smaller the difference between the side lengths of a rectangle, the biggest the area*". The instructor asked the PSMTs to evaluate the conjecture and prove whether it was correct.

Merve:(Drew three rectangles with the side lengths of 12 by 6, 15 by 3, and 9 by 9). It is true. These rectangles have the same perimeter, 36. But, the area of the square is bigger than the other two rectangles.

Instructor: Do you think that Merve proved Cihat's conjecture?

PST: No, she just demonstrated for those rectangles.

Instructor: What is missing in her argument?

PSTs: It is not general

Instructor: We mentioned that providing examples do not suffice as mathematical proofs. How many examples can I draw with a perimeter of 36?

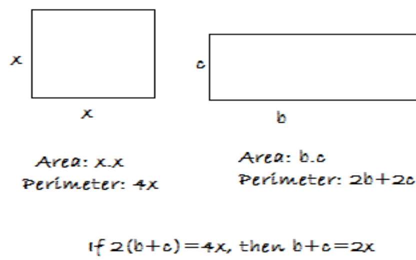
PSTs: 5? (Said as if they were asking if it was true). Infinitely many?

Cihat: Infinitely many, because in between whole numbers, there are infinitely many rational numbers

Instructor: So we can draw infinitely many rectangles with the perimeter of 36, will you be able to try all of these rectangles out like Merve attempted to do here?

PSTs: No!

Yılmaz: (Volunteered to share his argument). Now we have the lengths of b,c (referring to the long and short sides of a rectangle in this order) and x (referring to a side length of a square). They should have the following relationships: $b>x>c$. Thus, $x^2>b.c$. Let's assume that $x=n$ and $c=n-1$ and $b=n+1$. Therefore, $x^2=n^2>b.c=n^2-1$



Instructor: Why does b have to be bigger than x and x has to be bigger than c?

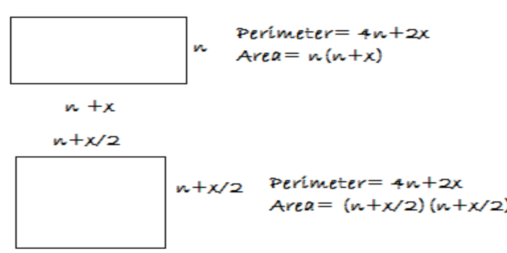
PSTs: If these two rectangles have the same perimeter, than this relationship should hold.

Instructor: Ok, but why should x between b and c?

Cihat: b and c should be different in lengths, because we consider the rectangles that are not squares, so $b\neq c$. Then, we know that $x=\frac{b+c}{2}$ since the perimeter of the two shapes should be equal. Thus x should be between b and c. We know that $x=\frac{b+c}{2}$so, $x^2=\frac{b^2+c^2+2bc}{4}$. We know that $b-c>0$, so $(b-c)^2>0.b^2-2bc+c^2>0 \Rightarrow b^2+c^2>2bc$. If $b^2+c^2>2bc$, Then $\frac{b^2+c^2+2bc}{4}$ should be bigger than b.c (the area of the rectangle).

Instructor: Ok, great.Zeynep, would you like to share your method with us?

Zeynep: (Writing her argument on the board).The perimeters of these rectangles should be the same.$A_1= n^2+nx$ and $A_2= n^2+nx+x^2/4$. Thus, it is obvious that A_2 should be bigger than A_1 since $x\neq 0$.



Aysegul: (Writing her argument on the board).

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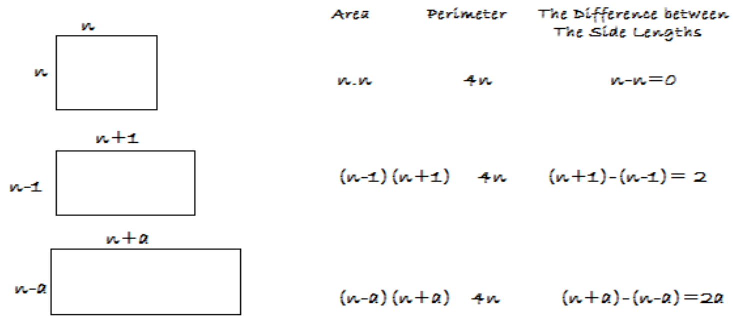

Cihat's conjecture was constructed as a response to the Instructor's request. Therefore, it was coded as Type C_1. Merve provided three examples that demonstrated that Cihat's conjecture was true. Since Merve used an invalid mode of argumentation-inductive argument-, her argument was coded as Type P_4. Stylianides (2007) argues that the main difference between empirical arguments and proofs lies in the modes of argumentation: invalid versus valid modes of argumentation. Empirical arguments provide inconclusive evidence by verifying the statement's truth only for a proper subset of all covered by the generalization, whereas proofs provide conclusive evidence truth by treating appropriately all cases covered by the generalization. When asked to evaluate the argument, both Merve and the other PSMTs in the class were able to state this limitation of the argument. Stylianides & Stylianides (2009) argued that construction-evaluation tasks can better identify prospective teachers' who seem to posses the empirical justification scheme. Unlike Merve, Yilmaz attempted to construct a deductive argument. However, his argument did not provide justification for some of the assertions he used in his argument (i.e. $b>x>c$). Additionally, Yilmaz's argument was constructed based on a condition- the side lengths of the rectangle and the square should be consecutive. Yilmaz's argument was coded as Type P_3. Cihatwas able to provide the justifications for each step of his argument. Thus, his argument as well as Zeynep's and Aysegul's arguments was coded as Type P_1.

Conclusion and Discussion

Given that teachers' ability to teach mathematics depends on the quality of their subjectmatter knowledge, a necessary condition for the realization of the importance of mathematical proofs as stated in the current curriculum reforms (NGA/CCSSO, 2010; NCTM,2000) is that teachers of all levels have good understanding of proofs (Stylianides & Ball, 2008). This study reported pre-service mathematics teachers' engagement with proof-and-conjecture tasks.The results of the study demonstrated that the number of instances where the PSMTs constructed conjectures, which isreferred as one of the essential parts of the process of making sense of and establishing mathematical knowledge (Stylianides, 2008), were limited and usually occurred when asked explicitly. Constructing arguments to prove and/or disprove assertions, on the other hand, occurred more often. Furthermore, the majority of the arguments constructed highlighted the explanatory aspect, which is consistent with the results of many studies that claimed that in mathematics classrooms, it would be more useful to use proof as a tool to explain than to convince (Hanna, 1990; Knuth, 2002).

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